Solution of Final MT101
January 2019

Prob. 1

a. Limits

i. $\lim\limits_{x\to 0} \dfrac{x(\cos x-1)}{\sin x - x} = \dfrac{0}{0}$ diff.

$\lim\limits_{x\to 0} \dfrac{1(\cos x-1)+x(-\sin x)}{\cos x - 1} = \lim\limits_{x\to 0} \dfrac{-x\sin x+\cos x-1}{\cos x-1} = \dfrac{0}{0}$

diff. $\lim\limits_{x\to 0} \dfrac{-x\cos x-\sin x-\sin x}{-\sin x} = \lim\limits_{x\to 0} \dfrac{x\cos x+2\sin x}{\sin x} = \dfrac{0}{0}$

diff. $\lim\limits_{x\to 0} \dfrac{\cos x - \cancel{x\sin x}^{0} + 2\cos x}{\cos x} = \dfrac{1+2}{1} = 3$

ii. $\lim\limits_{x\to 1} \dfrac{2x^2-(3x+1)\sqrt{x}+2}{x-1} = \dfrac{0}{0}$ diff.

$\lim\limits_{x\to 1} \dfrac{4x-\left[3*\sqrt{x}+(3x+1)\frac{1}{2}\frac{1}{\sqrt{x}}\right]}{1} = \dfrac{4-\left[3+4*\frac{1}{2}\right]}{1} = -1$

iii $\lim\limits_{x\to 0} (\cot x)^{\sin x} = \infty^0$

let $y = \lim\limits_{x\to 0} (\cot x)^{\sin x}$

$\ln y = \ln \lim\limits_{x\to 0} (\cot x)^{\sin x} = \lim\limits_{x\to 0} \ln(\cot x)^{\sin x}$

$= \lim\limits_{x\to 0} \sin x \ln \cot x = 0\cdot\infty$

$= \lim\limits_{x\to 0} \dfrac{\ln \cot x}{\csc x} = \dfrac{\infty}{\infty}$ diff.

$= \lim\limits_{x\to 0} \dfrac{\frac{1}{\cot x}\cdot -\csc^2 x}{-\csc x\cdot\cot x} = \lim\limits_{x\to 0} \dfrac{+\csc^2 x}{-\csc x\cdot\cot^2 x}$

$= \lim\limits_{x\to 0} \dfrac{\csc x}{\cot^2 x} = \lim\limits_{x\to 0} \dfrac{\tan^2 x}{\sin x}$

$= \lim\limits_{x\to 0} \dfrac{\sin^2 x}{\cos^2 x\cdot\cancel{\sin x}} = \lim\limits_{x\to 0} \dfrac{\sin x}{\cos^2 x} = \dfrac{0}{1} = 0$

$\ln y = 0 \qquad y = e^0 = 1$

$= \lim\limits_{x\to 0} (\cot x)^{\sin x} = 1$

iv. $\lim\limits_{x\to 1} (2-x)^{\tan\frac{\pi}{2}x} = 1^{\infty}$

let $y = \lim\limits_{x\to 1} (2-x)^{\tan\frac{\pi}{2}x}$

$\ln y = \ln \lim\limits_{x\to 1} (2-x)^{\tan\frac{\pi}{2}x} = \lim\limits_{x\to 1} \ln (2-x)^{\tan\frac{\pi}{2}x}$

$= \lim\limits_{x\to 1} \tan\frac{\pi}{2}x \cdot \ln(2-x) = \infty \cdot 0$

$= \lim\limits_{x\to 1} \dfrac{\ln(2-x)}{\cot\frac{\pi}{2}x}$ diff. $= \lim\limits_{x\to 1} \dfrac{\frac{1}{2-x}\cdot(-1)}{-\frac{\pi}{2}\csc^2\frac{\pi}{2}x}$

$= \lim\limits_{x\to 1} \dfrac{\sin^2\frac{\pi}{2}x}{(2-x)\frac{\pi}{2}} = \dfrac{1}{\frac{\pi}{2}} = \dfrac{2}{\pi}$

$\therefore \ln y = \frac{2}{\pi}$

$y = e^{\frac{2}{\pi}} \longrightarrow \lim\limits_{x\to 1} (2-x)^{\tan\frac{\pi}{2}x} = e^{\frac{2}{\pi}} \longrightarrow$

b. $f(x) = \begin{cases} x^2 & x < -2 \\ x+6 & x \geq -2 \end{cases}$

at $x = -2$ $\quad f(x) = -2+6 = 4$

$\lim\limits_{x\to -2^-} f(x) = (-2)^2 = 4$

$\lim\limits_{x\to -2^+} f(x) = -2+6 = 4$

$f(x)$ is continuous at $x = -2$

c. $f(x) = 3x-2$

to get $f^{-1}(x)$ let $y = 3x-2$

$\longrightarrow x = \frac{1}{3}(y+2)$

$\therefore f^{-1}(x) = \frac{1}{3}(x+2) \longrightarrow$

$(f \circ f^{-1})(x) = 3\left[\frac{1}{3}(x+2)\right] - 2 = x+2-2 = x$

Prob (2)

a. diff.

i. $y = \tan\left(\sqrt[3]{3x^2} + \ln(5x^4)\right)$

$y' = \frac{dy}{dx} = \sec^2\left(\sqrt[3]{3x^2} + \ln(5x^4)\right)\left[\frac{1}{3}(3x^2)^{-\frac{2}{3}} \times 6x + \frac{20x^3}{5x^4}\right]$

$= \left(2x \cdot (3x^2)^{-\frac{2}{3}} + \frac{4}{x}\right) \cdot \sec^2\left(\sqrt[3]{3x^2} + \ln(5x^4)\right) \longrightarrow$

ii. $x^3 y^5 + 3x = 8y^3 + 1$

diff. $3x^2 \cdot y^5 + 5x^3 y^4 \cdot y' + 3 = 24y^2 \cdot y'$

$3x^2 y^5 + 3 = (24y^2 - 5x^3 y^4)\, y'$

$\therefore y' = \frac{dy}{dx} = \frac{3x^2 y^5 + 3}{24y^2 - 5x^3 y^4} \longrightarrow$

iii $y = x^{x^{x^4}}$

$\ln y = x^{x^4} \ln x$

$\ln(\ln y) = \ln(x^{x^4} \cdot \ln x) = \ln x^{x^4} + \ln(\ln x)$

$= x^4 \ln x + \ln(\ln x)$

diff. $\frac{1}{\ln y} \cdot \frac{1}{y} \cdot y' = x^4 \cdot \frac{1}{x} + 4x^3 \ln x + \frac{1}{\ln x} \cdot \frac{1}{x}$

$y' = y \cdot \ln y \left[x^3 + 4x^3 \ln x + \frac{1}{x \ln x}\right]$

$= x^{x^{x^4}} \cdot \ln(x^{x^{x^4}}) \cdot \left[x^3 + 4x^3 \ln x + \frac{1}{x \ln x}\right]$

b. $y = e^{1-2x^3}$

$y' = -6x^2 \cdot e^{1-2x^3}$

$y'' = -12x \cdot e^{1-2x^3} - 6x^2 \cdot (-6x^2) e^{1-2x^3}$

$= e^{1-2x^3}(36x^4 - 12x)$

c. $y = (5x+3)^4$

$y' = 4(5x+3)^3 \cdot 5 = 20(5x+3)^3$

$y'' = 20 \times 3(5x+3)^2 \times 5 = 300(5x+3)^2$

$y''' = 300 \times 2\,(5x+3) \times 5 = 3000\,(5x+3)$

$y^{(4)} = \frac{d^4y}{dx^4} = 3000 \times 5 = 15000.$ $\longrightarrow$

Prob. 3

a. $f(x) = \sqrt{x-1}$

$f(x)$ is contin. and diff. on $[1,\infty)$, so $f(x)$ is contin. & diff. on $[1,3]$

$f'(x) = \frac{1}{2\sqrt{x-1}}$ $\qquad f(a) = f(1) = \sqrt{0} = 0$

$f'(c) = \frac{1}{2\sqrt{c-1}}$ $\qquad f(b) = f(3) = \sqrt{3-1} = \sqrt{2}$

M.V. Thm $\quad f'(c) = \frac{f(b) - f(a)}{b-a}$

$\frac{1}{\cancel{2}\sqrt{c-1}} = \frac{\sqrt{2}-0}{\cancel{2}}$

$\therefore \sqrt{c-1} = \frac{1}{\sqrt{2}} \qquad c-1 = \frac{1}{2} \qquad c = \frac{3}{2} \in [1,3]$

b. $f(x) = 2x^3 - 3x^2 - 36x$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

For $f'(x) = 0 \qquad x^2 - x - 6 = 0 \qquad (x+2)(x-3) = 0$

$x = -2, 3$ critical points

$f''(x) = 12x - 6$

Point 1 $\quad x = -2$, $f(x) = 2(-2)^3 - 3(-2)^2 - 36(-2) = 44$

Point 2 $\quad x = 3 \quad f(x) = 2(3)^3 - 3(3)^2 - 36(3) = -81$

– at Point 1 $\quad f''(x) = 12(-2) - 6 = -30 \quad (-ve)$

$\therefore$ Point 1 $(-2, 44)$ is a local maximum

– at Point 2 $\quad f''(x) = 12 \times 3 - 6 = 30 \quad (+ve)$

$\therefore$ Point 2 $(3, -81)$ is a local minimum

c. $y = f(x)\, g(x)$

$f(x) = x^3, \quad g(x) = e^{-5x}$

$f'(x) = 3x^2 \qquad g'(x) = -5e^{-5x}$

$f''(x) = 6x \qquad g''(x) = (-5)^2 e^{-5x}$

$f'''(x) = 6 \qquad \vdots$

$f^{(4)}(x) = 0 \qquad g^{(n)}(x) = (-5)^n e^{-5x}$

From <u>Leibnitz Thm</u>

$$y_n = f g_n + n f_1 g_{n-1} + \frac{n(n-1)}{2!} f_2 g_{n-2} + \frac{n(n-1)(n-2)}{3!} f_3 g_{n-3} + \cdots + f_n g$$

sub.

$$y_n = x^3 (-5)^n e^{-5x} + n \cdot 3x^2 \cdot (-5)^{n-1} \cdot e^{-5x} + \frac{n(n-1)}{2!} (6x) (-5)^{n-2} e^{-5x} + \frac{n(n-1)(n-2)}{3!} \cdot 6 \cdot (-5)^{n-3} \cdot e^{-5x}$$

$$= e^{-5x} * \left[(-5)^n x^3 + 3n(-5)^{n-1} x^2 + 3n(n-1)(-5)^{n-2} \cdot x + n(n-1)(n-2)(-5)^{n-3} \right]$$

d. $f(x) = \sinh x \qquad f(0) = 0$

$f'(x) = \cosh x \qquad f'(0) = 1$

$f''(x) = \sinh x \qquad f''(0) = 0$

$f'''(x) = \cosh x \qquad f'''(0) = 1$

$$\sinh x \stackrel{!}{=} f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

Prob. 4

a. $\frac{4x^4+11x^2-4x+4}{x(x^2+2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$

$$4x^4+11x^2-4x+4 = A(x^2+2)^2 + (Bx+C)x(x^2+2) + (Dx+E)x$$

x^0: $4 = 4A$ $\quad A = 1$

coeff^{n} of x^3: $0 = C$ $\quad C = 0$

coeff^{n} of x^4: $4 = A + B$ $\quad \therefore B = 4 - A = 4 - 1 = 3$

coeff^{n} of x^2: $11 = 4A + 2B + D$

$11 = 4\times1 + 2\times3 + D$

$\therefore D = 11 - 4 - 6 = 1$

coeff^{n} of x $\quad -4 = 2C + E$

$E = -4 - 2C = -4$

$$\therefore \frac{4x^4+11x^2-4x+4}{x(x^2+2)^2} = \frac{1}{x} + \frac{3x}{x^2+2} + \frac{x-4}{(x^2+2)^2} \longrightarrow$$

b. step 1 at $n=1$

$1^3 = 1 = \left(\frac{1\times2}{2}\right)^2 = 1$ $\quad$ relation is true for $n=1$

step 2 at $n=k$

let $1^3+2^3+3^3+\dots+k^3 = \left(\frac{k(k+1)}{2}\right)^2$

step 3 put $n=k+1$

$1^3+2^3+3^3+\dots+k^3+(k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$

R.H.S. $= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left[\left(\frac{k}{2}\right)^2 + (k+1)\right] \times \frac{4}{4}$

$= \left(\frac{(k+1)}{2}\right)^2 (k^2+4(k+1)) = \frac{(k+1)^2}{2^2}(k^2+4k+4)$

$= \frac{(k+1)^2}{2^2}(k+2)^2 = \left(\frac{(k+1)(k+2)}{2}\right)^2 \longrightarrow$

Then the relation is true of $n=k+1$

the " " " for all $n \geq 1$

c. $\left(x - \frac{3}{x}\right)^9$

i^{th} term $\binom{n}{i-1} a^{n-i+1} \cdot b^{i-1}$

$n = 9$, $a = x$, $b = -3x^{-1}$

i^{th} term $= \binom{9}{i-1} x^{9-i+1} (-3x^{-1})^{i-1}$

$= \binom{9}{i-1} (-3)^{i-1} \cdot x^{10-i-i+1}$

$= \binom{9}{i-1} (-3)^{i-1} (x)^{11-2i}$

$11 - 2i = 5 \qquad i = 3 \longrightarrow$ third term

3^{rd} term $= \binom{9}{2} x^5 (-3)^2 = \frac{9!}{2!\,7!} x^5 \times 9$

$= \frac{9 \times \cancel{8}^{4} \times \cancel{7!}}{\cancel{2} \times \cancel{7!}} \cdot 9\, x^5 = 324\, x^5 \longrightarrow$

Prob. 5

a. $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & -1 & 4 \\ 5 & 9 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & -2 \\ 1 & -2 & 4 \end{bmatrix}$

$A \times B = \begin{bmatrix} 21 & 3 & 16 \\ 17 & -5 & 21 \\ 34 & 20 & 1 \end{bmatrix} \qquad (A \times B)^T = \begin{bmatrix} 21 & 17 & 34 \\ 3 & -5 & 20 \\ 16 & 21 & 1 \end{bmatrix} \longrightarrow 1$

$B^T = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 3 & -2 \\ 1 & -2 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 2 & 3 & 5 \\ 3 & -1 & 9 \\ 5 & 4 & 1 \end{bmatrix}$

$B^T \times A^T = \begin{bmatrix} 21 & 17 & 34 \\ 3 & -5 & 20 \\ 16 & 21 & 1 \end{bmatrix} \longrightarrow 2$

from 1, 2 $\quad \therefore (A \times B)^T = B^T \times A^T$

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b. $\det|G| = 0$ for singular matrix

$\det(G) = 3(7C - 45) - 2(4C - 9) + 5(20 - 7)$

$0 = 21C - 135 - 8C + 18 + 65 = 13C - 52$

$\therefore C = \frac{52}{13} = 4 \longrightarrow$

c. $\left[\begin{array}{ccc|c} 3 & 1 & -2 & -7 \\ 2 & 2 & 1 & 9 \\ -1 & -1 & 3 & 6 \end{array}\right] \xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 2 & 2 & 1 & 9 \\ -1 & -1 & 3 & 6 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 + R_1 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 4 & 7 & 41 \\ 0 & -2 & 0 & -10 \end{array}\right]$

$\xleftarrow{R_3 \leftrightarrow R_2}$

$\left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & -2 & 0 & -10 \\ 0 & 4 & 7 & 41 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 1 & 0 & 5 \\ 0 & 4 & 7 & 41 \end{array}\right] \xrightarrow{R_3 - 4R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 7 & 21 \end{array}\right]$

$\xrightarrow{\frac{1}{7}R_3} \left[\begin{array}{ccc|c} 1 & -1 & -3 & -16 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 + 3R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -7 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Sol^n: $x = -2$, $y = 5$, $z = 3$